Normal and Oblique shoks

Shock wave :

The compression wave occurs infront of the bodies in Supersonic Flow.

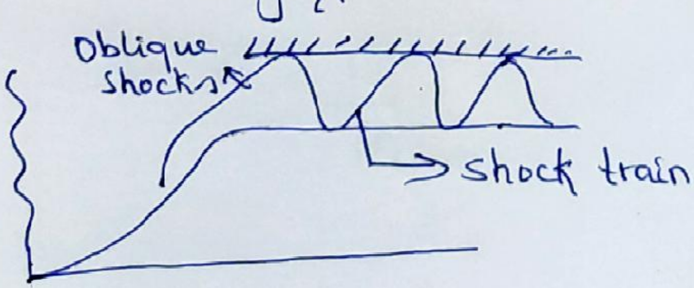
Types :

- Normal shock
- Oblique shock
- Expansion Waves

Shock Train :

The shock is allowed to get reflected inside the duct is called Shock train or shock reflection.

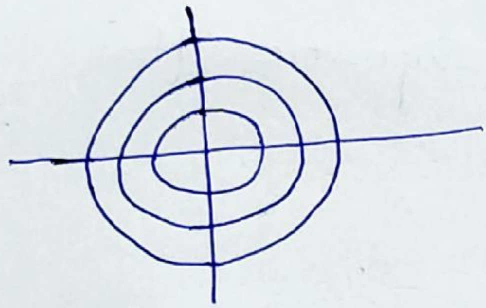
This highly increases the Mach number.



Formation of Normal shock

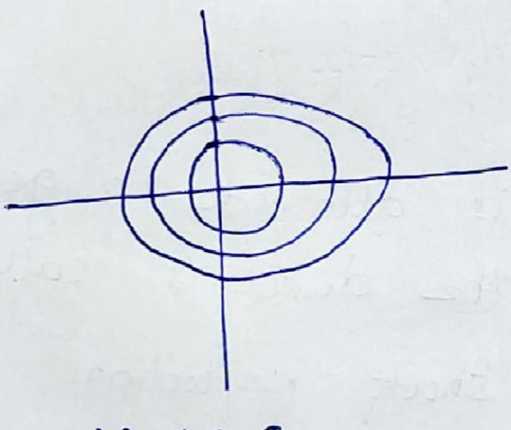
Wave front :-

i) For Incompressible Flow



$M < 0.3$

ii) For Subsonic



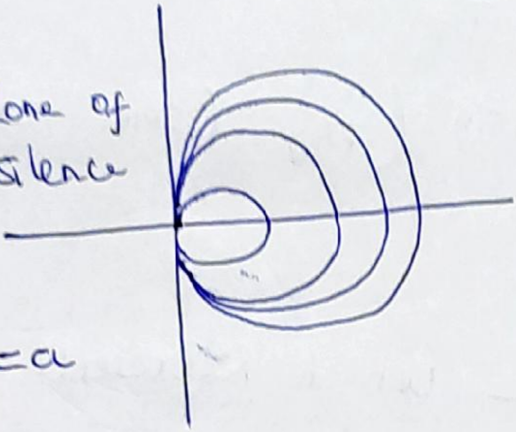
$u = 0.7a$

$u \rightarrow$ Flow velocity

$a \rightarrow$ acoustic velocity (or) Velocity of sound

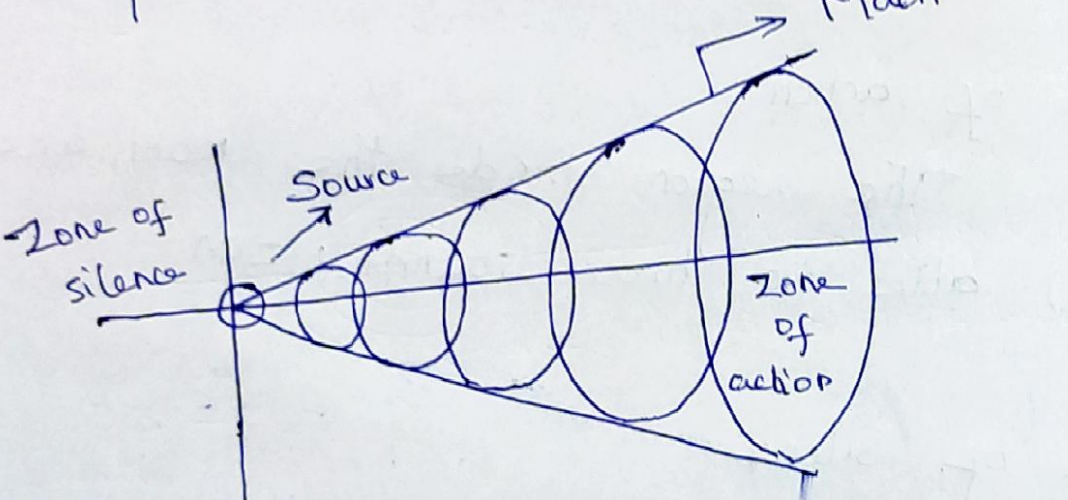
$0.3 < M < 0.8$

iii) For Sonic



$u = a$

For
iv) Supersonic :



$M > 1.2$

$$\sin \alpha = \frac{1}{M}$$

Conical angle or Mach angle:

$$\sin \alpha = \frac{1}{M}$$

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right)$$ (semi-conical angle)

Mach cone:

The cone which represents the source and sound travelling is called Mach cone.

Zone of action:

The region inside the mach cone (i.e) able to hear sounds, ZOA.

Zone of Silence:

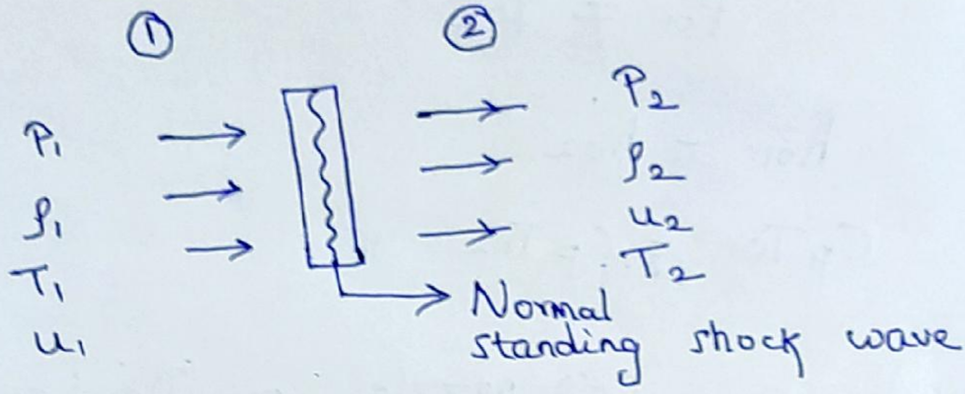
The region outside the mach cone (i.e) Can't able to hear sounds, ZOS.

Road Map for Unit II :

- Normal Shocks
  - i) Prandtl Relation
  - ii) Relation between M_1 and M_2
  - iii) Relations for $\frac{P_2}{P_1}$, $\frac{\rho_2}{\rho_1}$, $\frac{T_2}{T_1}$, $\frac{P_{02}}{P_{01}}$
  - iv) Rankine – Hugoneit relation
- Oblique shocks
  - i) θ-β-M relation
  - ii) Hodograph and pressure turning angle
  - iii) Shock polar.

Normal Shocks – Basic relations :

Consider a shock as shown in the fig.



① → Represents before shock

② → Represents after the shock

Consider the properties before shock is

u_1 → Flow velocity before shock

P_1 → Pressure before shock

T_1 → Temperature before shock

ρ_1 → Density before the shock

Similarly u_2, P_2, T_2, ρ_2 for after the shocks

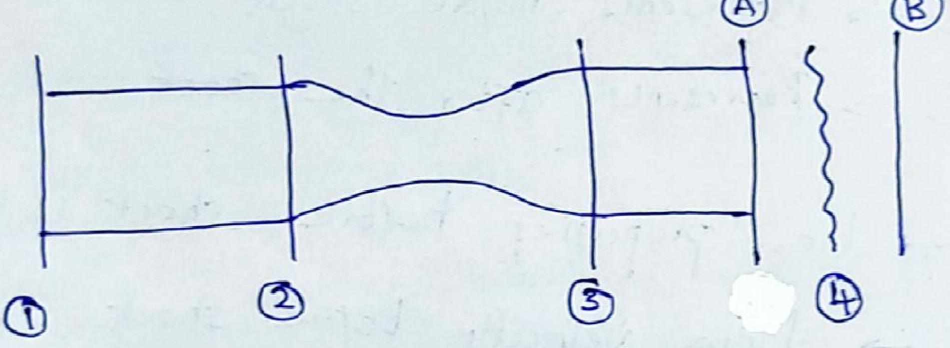
The pressure P_{01} and P_{02} is changed from P_{01} to P_{02}

$$P_{01} \neq P_{02}$$

$$h_{01} = h_{02}$$

$$C_p T_{01} = C_p T_{02}$$

Consider a C-D nozzle or De-Laval nozzle:



① - Ⓐ ⇒ Isentropic region

Ⓐ - Ⓑ ⇒ Normal shock region.

Condition :.

$$P_{01} = P_{02} = P_{03} \neq P_{04}$$

$$T_{01} = T_{02} = T_{03} = T_{04}$$

By continuity eqn

$$\dot{m} = \rho_1 A_1 u_1 = \rho_2 A_2 u_2$$

$$\rho_1 u_1 = \rho_2 u_2 = \frac{\dot{m}}{A}$$

Mass flux, $\frac{\dot{m}}{A} = \rho_1 u_1 = \rho_2 u_2 \rightarrow (1)$

By momentum eqn :.

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

$$P_1 - P_2 = \rho_2 u_2^2 - \rho_1 u_1^2$$

$$P_1 - P_2 = \rho_2 u_2 u_2 - \rho_1 u_1 u_1$$

$$= \frac{\dot{m}}{A} u_2 - \frac{\dot{m}}{A} u_1$$

$$P_1 - P_2 = \frac{\dot{m}}{A}(u_2 - u_1) \rightarrow (2)$$

Energy eqn interms of Normal Shock waves

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$C_pT_1 + \frac{u_1^2}{2} = C_pT_2 + \frac{u_2^2}{2}$$

$$\frac{\gamma R}{\gamma-1} T_1 + \frac{u_1^2}{2} = \frac{\gamma R}{\gamma-1} T_2 + \frac{u_2^2}{2}$$

$$\frac{a_1^2}{\gamma-1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma-1} + \frac{u_2^2}{2}$$

$$\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma-1} + \frac{a^{*2}}{2}$$

$$\frac{a^2}{\gamma-1} - \frac{u_1^2}{2} = a^{*2}\left(\frac{1}{\gamma-1} + \frac{1}{2}\right)$$

$$= a^{*2}\frac{(2+\gamma-1)}{2(\gamma-1)}$$

$$= \frac{a^{*2}(\gamma+1)}{2(\gamma-1)}$$

$$\frac{a^2}{\gamma-1} - \frac{u^2}{2} = \frac{a^{*2}}{2}\left(\frac{\gamma+1}{\gamma-1}\right)$$

For isentropic process

$$\left(\frac{\partial p}{\partial \rho}\right)_s = a^2$$

$$p = C\rho^\gamma \text{ (from velocity of sound)}$$

$$\frac{\partial p}{\partial \rho} = C\gamma\rho^{\gamma-1}$$

$$= C\gamma\rho^\gamma\rho^{-1}$$

$$\frac{\partial p}{\partial \rho} = \frac{C\gamma\rho^\gamma}{\rho}$$

if $P = C\rho^\gamma$, $C = \frac{P}{\rho^\gamma}$

$$\frac{\partial p}{\partial \rho} = \frac{P}{\rho^\gamma} \times \gamma \times \frac{\rho^\gamma}{\rho}$$

$$\frac{\partial p}{\partial \rho} = \frac{\gamma p}{\rho}$$

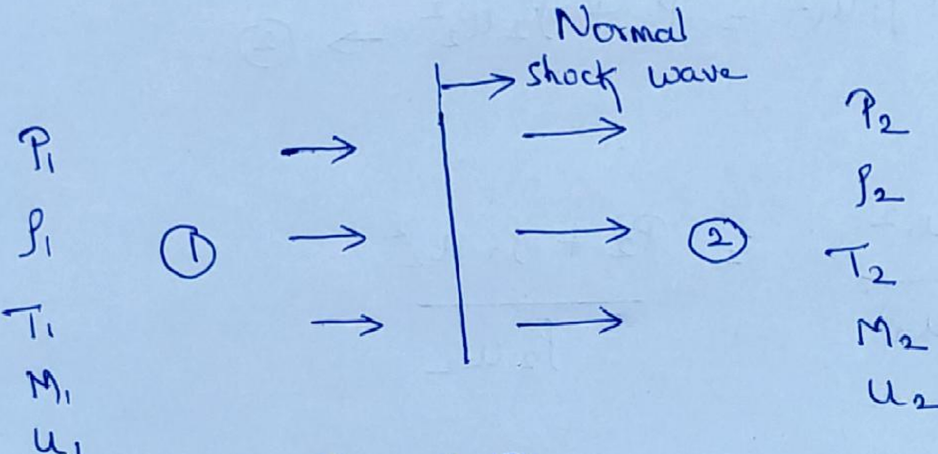
$$\left(\frac{\partial p}{\partial \rho}\right)_s = a^2$$

$$a^2 = \frac{\gamma p}{\rho}$$

Prandtl - Relation for Normal shocks :

Physics :

Consider a normal standing shock wave inside a duct with flow properties before shock are $M_1, u_1, P_1, \rho_1, T_1$ & P_{01} and after the shock are $M_2, u_2, P_2, \rho_2, T_2$ & P_{02}.



$$P_{01} \neq P_{02}$$

$$T_{01} = T_{02}$$

Concept :

Combination of Continuity, energy and momentum equation.

To derive :

$$u_1 u_2 = a^{*2}$$

$$1 = M_1^* M_2^*$$

By continuity eqn:

$$\rho_1 A_1 u_1 = \rho_2 A_2 u_2$$

$$A_1 = A_2$$

$$\rho_1 u_1 = \rho_2 u_2 \quad \text{— ①}$$

By Momentum eqn:

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \rightarrow ②$$

② ÷ ①

$$\frac{P_1 + \rho_1 u_1^2}{\rho_1 u_1} = \frac{P_2 + \rho_2 u_2^2}{\rho_2 u_2}$$

$$\frac{P_1}{\rho_1 u_1} + u_1 = \frac{P_2}{\rho_2 u_2} + u_2 \quad \text{— ③}$$

$$a = \sqrt{\gamma RT}$$

$$P = \rho RT$$

$$\frac{P}{\rho} = RT$$

$$a = \sqrt{\frac{\gamma P}{\rho}}$$

$$a^2 = \frac{\gamma P}{\rho}$$

$$\frac{a^2}{\gamma} = \frac{P}{\rho}$$

$$\frac{a_1^2}{\gamma u_1} + u_1 = \frac{a_2^2}{\gamma u_2} + u_2$$

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1 \quad \text{--- (4)}$$

By energy eqn :

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$h = C_p T$$

$$C_p T_1 + \frac{u_1^2}{2} = C_p T_2 + \frac{u_2^2}{2}$$

$$C_p = \frac{\gamma R}{\gamma - 1}$$

$$\frac{\gamma R}{\gamma - 1} T_1 + \frac{u_1^2}{2} = \frac{\gamma R}{\gamma - 1} T_2 + \frac{u_2^2}{2}$$

$$\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2} \rightarrow (5)$$

$$a_2 = a^*$$

$$u_2 = a^*$$

$$\frac{a_1^2}{\gamma-1} + \frac{u_1^2}{2} = \frac{a^{*2}}{\gamma-1} + \frac{a^{*2}}{2}$$

$$\frac{a_1^2}{\gamma-1} + \frac{u_1^2}{2} = \frac{2a^{*2} + a^{*2}(\gamma-1)}{2(\gamma-1)}$$

$$= \frac{a^{*2}(2+\gamma-1)}{2(\gamma-1)}$$

$$\frac{a_1^2}{\gamma-1} + \frac{u_1^2}{2} = \frac{a^{*2}(\gamma+1)}{2(\gamma-1)}$$

$$\frac{a_1^2}{\gamma-1} = \frac{a^{*2}(\gamma+1)}{2(\gamma-1)} - \frac{u_1^2}{2}$$

$$a_1^2 = \frac{a^{*2}(\gamma+1)(\cancel{\gamma-1})}{2(\cancel{\gamma-1})} - \frac{u_1^2}{2}(\gamma-1)$$

$$a_1^2 = \frac{a^{*2}(\gamma+1)}{2} - \frac{u_1^2}{2}(\gamma-1) \quad \text{— (6)}$$

Similarly

$$a_2^2 = \frac{a^{*2}(\gamma+1)}{2} - \frac{u_2^2}{2}(\gamma-1) \quad \text{— (7)}$$

Sub ⑥ & ⑦ in ④

$$\frac{\frac{a^{*2}(\gamma+1)}{2} - \frac{u_1^2}{2}(\gamma-1)}{\gamma u_1} - \left(\frac{\frac{a^{*2}(\gamma+1)}{2} - \frac{u_2^2}{2}(\gamma-1)}{\gamma u_2} \right) = u_2 - u_1$$

$$\frac{a^{*2}(\gamma+1)}{2\gamma u_1} - \frac{u_1^2(\gamma-1)}{2\gamma u_1} - \frac{a^{*2}(\gamma+1)}{2\gamma u_2} + \frac{u_2^2(\gamma-1)}{2\gamma u_2} = u_2 - u_1$$

$$\frac{a^{*2}(\gamma+1)}{2\gamma}\left(\frac{1}{u_1} - \frac{1}{u_2}\right) + \frac{(\gamma-1)}{2\gamma}(u_2 - u_1) = u_2 - u_1$$

$$\frac{a^{*2}(\gamma+1)}{2\gamma}\left(\frac{u_2 - u_1}{u_1 u_2}\right) + \frac{(\gamma-1)}{2\gamma}(u_2 - u_1) = u_2 - u_1$$

$$\frac{a^{*2}(\gamma+1)}{2\gamma u_1 u_2}(u_2 - u_1) + \frac{(\gamma-1)(u_2 - u_1)}{2\gamma} = u_2 - u_1$$

(with $(u_2 - u_1)$ struck out)

$$\frac{a^{*2}(\gamma+1)}{2\gamma u_1 u_2} + \frac{(\gamma-1)}{2\gamma} = 1$$

$$\frac{a^{*2}(\gamma+1)}{2\gamma u_1 u_2} = 1 - \frac{(\gamma-1)}{2\gamma}$$

$$= \frac{2\gamma - \gamma + 1}{2\gamma}$$

$$\frac{a^{*2}(\gamma+1)}{2\gamma u_1 u_2} = \frac{\gamma+1}{2\gamma}$$

$$\frac{a^{*2}}{u_1 u_2} = 1$$

$$a^{*2} = u_1 u_2 \rightarrow (8)$$

The square of critical acoustic velocity is equal to the product of flow velocities before and after the shock.

$$a^{*2} = u_1 u_2 \Rightarrow 1 = \frac{u_1}{a^*} \frac{u_2}{a^*}$$

$$1 = M_1^* M_2^* \rightarrow (9)$$

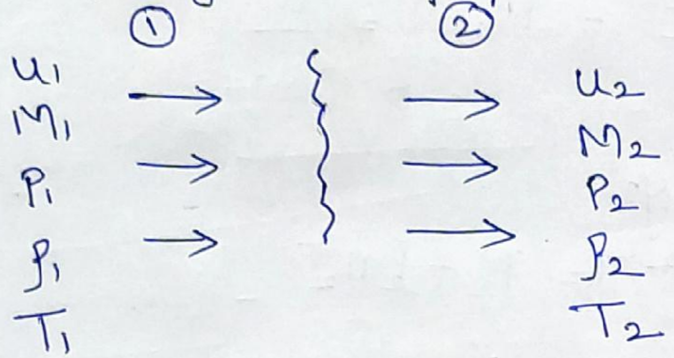
The product of critical Mach number before and after the shock always gives unity.

Eqn (8) & (9) are Prandtl relations for Normal shocks.

Static Pressure ratio, Density ratio, temperature ratio and stagnation pressure ratio, across the shock :.

Physics :.

Consider a normal shock inside a duct with corresponding flow properties.



Concept :.

Combination of continuity equation and Momentum equation.

By continuity eqn :.

$$\rho_1 A_1 u_1 = \rho_2 A_2 u_2$$

$$A_1 = A_2$$

$$\rho_1 u_1 = \rho_2 u_2$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2}$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \times \frac{u_1}{u_1}$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1^2}{u_1 u_2} = \frac{u_1^2}{a^{*2}} = M_1^{*2}$$

$$\frac{\rho_2}{\rho_1} = M_1^{*2} = \frac{u_1}{u_2}$$

By momentum eqn:

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

$$P_2 - P_1 = \rho_1 u_1^2 - \rho_2 u_2^2$$

WKT,

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_2 = \frac{\rho_1 u_1}{u_2}$$

$$P_2 - P_1 = \rho_1 u_1^2 - \frac{\rho_1 u_1}{u_2} u_2^2$$

$$P_2 - P_1 = \rho_1 u_1^2 - \rho_1 u_1 u_2$$

$$P_2 - P_1 = \rho_1 u_1 (u_1 - u_2)$$

Divide by 'P_1' on both the sides

$$\frac{P_2 - P_1}{P_1} = \frac{\rho_1 u_1}{P_1} (u_1 - u_2)$$

$PV = mRT$, $P = \frac{m}{v} RT \Rightarrow P = \rho RT$

$$\frac{\rho}{P} = \frac{1}{RT}$$

$a^2 = \gamma RT$, $\frac{a^2}{\gamma} = RT$,

$$\frac{\rho}{P} = \frac{\gamma}{a^2}$$

$$\frac{P_2 - P_1}{P_1} = \frac{\gamma}{a_1^2} u_1 (u_1 - u_2)$$

$$\frac{P_2 - P_1}{P_1} = \frac{\gamma}{a_1^2} u_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

$$\frac{P_2}{P_1} - 1 = \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

$$\frac{P_2}{P_2} = 1 + \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right) \quad \text{--- (1)}$$

WKT,

$$\frac{\rho_2}{\rho_1} = M_1^{*2} = \frac{u_1}{u_2}$$

$$M_1^{*2} = \frac{u_1}{u_2} \Rightarrow \frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2} = \frac{u_1}{u_2} \quad \text{--- (2)}$$

Sub. (2) in (1)

$$\frac{P_2}{P_1} = 1 + \gamma M_1^2 \left[1 - \frac{[2 + (\gamma - 1) M_1^2]}{(\gamma+1) M_1^2} \right]$$

$$\frac{P_2}{P_1} = 1 + \gamma M_1^2 \left\{ \frac{(\gamma+1) M_1^2 - 2 - (\gamma-1) M_1^2}{(\gamma+1) M_1^2} \right\}$$

$$\frac{P_2}{P_1} = 1 + \gamma \left[\frac{\gamma M_1^2 + M_1^2 - 2 - \gamma M_1^2 + M_1^2}{(\gamma+1)} \right]$$

$$\frac{P_2}{P_1} = 1 + \gamma \left[\frac{2 M_1^2 - 2}{(\gamma+1)} \right]$$

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma (M_1^2 - 1)}{(\gamma+1)} \rightarrow (3)$$

Eqn (3) is Known as static pressure ratio across the shock.

$$P = \rho R T$$

$$P_1 = \rho_1 R T_1$$

$$P_2 = \rho_2 R T_2$$

$$\frac{P_2}{P_1} = \frac{\rho_2 R T_2}{\rho_1 R T_1}$$

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)\left(\frac{\rho_1}{\rho_2}\right)$$

$$\frac{T_2}{T_2} = \left[1 + \frac{2\gamma(M_1^2 - 1)}{(\gamma+1)}\right]\left[\frac{2 + (\gamma+1) M_1^2}{(\gamma+1) M_1^2}\right] \rightarrow (4)$$

Eqn (4) is known as temperature ratio, across the shock.

$$h = C_p T$$

$$h_1 = C_p T_1$$

$$h_2 = C_p T_2$$

$$\frac{h_2}{h_1} = \frac{C_p T_2}{C_p T_1}$$

$$\frac{h_2}{h_1} = \frac{T_2}{T_1} = \left[1 + \frac{2\gamma(M_1^2 - 1)}{(\gamma+1)}\right]\left[\frac{2 + (\gamma+1) M_1^2}{(\gamma+1) M_1^2}\right]$$

WKT

$$M_2^2 = \frac{2 + (\gamma-1) M_1^2}{2\gamma M_1^2 - (\gamma-1)}$$

$$M_1 = f(M_2) = f\left(\frac{P_2}{P_1}\right) = f\left(\frac{T_2}{T_1}\right) = f\left(\frac{h_2}{h_1}\right)$$
$$= f\left(\frac{\rho_2}{\rho_1}\right)$$

WKT

$$S_2 - S_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{P_2}{P_1}\right) \quad — (5)$$

$$S_2 - S_1 = C_p \ln\left[\left(1 + \frac{2\gamma(M_1^2-1)}{(\gamma+1)}\right)\left(\frac{2 + (\gamma-1) M_1^2}{(\gamma+1) M_1^2}\right)\right]$$
$$- R \ln\left[1 + \frac{2\gamma(M_1^2-1)}{(\gamma+1)}\right] \rightarrow (6)$$

For stagnation state,

$$S_2 - S_1 = C_p \ln\left(\frac{T_{02}}{T_{01}}\right) - R \ln\left(\frac{P_{02}}{P_{01}}\right)$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$h_{01} = h_{02}$$

$$\cancel{C_p} T_{01} = \cancel{C_p} T_{02}$$

$$\frac{T_{02}}{T_{01}} = 1$$

$$S_2 - S_1 = C_p \ln(1) - R \ln\left(\frac{P_{02}}{P_{01}}\right) \rightarrow (7)$$

$$S_2 - S_1 = -R \ln\left(\frac{P_{02}}{P_{01}}\right) \rightarrow 8$$

$$\frac{S_2 - S_1}{R} = -\ln\left(\frac{P_{02}}{P_{01}}\right)$$

$$e^{\frac{(S_2 - S_1)}{R}} = -\frac{P_{02}}{P_{01}}$$

$$(5) = (8)$$

$$C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{P_2}{P_1}\right) = -R \ln\left(\frac{P_{02}}{P_{01}}\right)$$

$\div -R$

$$-\frac{C_p}{R} \ln\left(\frac{T_2}{T_1}\right) + \ln\left(\frac{P_2}{P_1}\right) = +\ln\left(\frac{P_{02}}{P_{01}}\right)$$

$$\ln\left(\frac{T_2}{T_1}\right)^{-\frac{C_p}{R}} + \ln\left(\frac{P_2}{P_1}\right) = \ln\left(\frac{P_{02}}{P_{01}}\right)$$

$$\ln\left[\left(\frac{T_2}{T_1}\right)^{-\frac{C_p}{R}}\left(\frac{P_2}{P_1}\right)\right] = \ln\left(\frac{P_{02}}{P_{01}}\right)$$

Taking 'e' power on both the sides

$$\left(\frac{T_2}{T_1}\right)^{-\frac{C_p}{R}} \left(\frac{P_2}{P_1}\right) = \frac{P_{02}}{P_{01}}$$

$$\frac{P_{02}}{P_{01}} = \left[\left(1 + \frac{2\gamma\,(M_1^2 - 1)}{(\gamma+1)}\right)\left(\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right)\right]^{-\frac{\gamma R}{(\gamma-1)R}} \times \left(1 + \frac{2\gamma\,(M_1^2 - 1)}{(\gamma+1)}\right)$$

$$\frac{P_{02}}{P_{01}} = \left(1 + \frac{2\gamma\,(M_1^2 - 1)}{(\gamma+1)}\right)^{-\frac{\gamma}{\gamma-1}} \left(\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right)^{-\frac{\gamma}{\gamma-1}} \left(1 + \frac{2\gamma\,(M_1^2 - 1)}{(\gamma+1)}\right)$$

$$\frac{P_{02}}{P_{01}} = \left(1 + \frac{2\gamma\,(M_1^2 - 1)}{(\gamma+1)}\right)^{-\frac{\gamma}{\gamma-1}+1} \left(\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right)^{-\frac{\gamma}{\gamma-1}}$$

$$-\frac{\gamma}{\gamma-1} + 1 = \frac{-\gamma + \gamma - 1}{\gamma - 1} = -\frac{1}{\gamma-1}$$

$$\frac{P_{02}}{P_{01}} = \left[1 + \frac{2\gamma\,(M_1^2 - 1)}{(\gamma+1)}\right]^{-\frac{1}{\gamma-1}} \left[\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right]^{-\frac{\gamma}{\gamma-1}}$$

Rankine – Hugonoit eqn :

(or)

Hugonit - eqn :

(or)

Mass motion velocity eqn :

It relates internal energy with pressure ratio (Pressure change and density change)

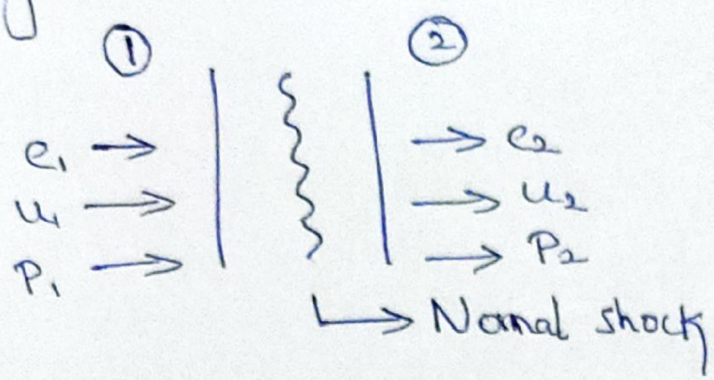
Significance :

It is used to relate thermodynamic quantities across the normal shocks without the reference of Mach number

Concept :

Combination of Steady state energy eqn momentum and Continuity eqns.

Physics:



By continuity eqn:

$$\dot{m}_1 = \dot{m}_2$$

$$\rho_1 A_1 u_1 = \rho_2 A_2 u_2$$

$$A_1 = A_2$$

$$\rho_1 u_1 = \rho_2 u_2 \quad \text{—①}$$

$$u_1 = \frac{\rho_2 u_2}{\rho_1} \quad \text{—②}$$

$$u_2 = \frac{\rho_1 u_1}{\rho_2} \quad \rightarrow ③$$

By momentum eqn:

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \quad \text{—④}$$

Sub (3) in (4)

$$P_1 + \rho_1 \frac{\rho_2^2 u_2^2}{\rho_1^2} = P_2 + \rho_2 u_2^2$$

$$P_2 - P_1 = \frac{\rho_2^2 u_2^2}{\rho_1} - \rho_2 u_2^2$$

$$P_2 - P_1 = \rho_2 u_2^2 \left(\frac{\rho_2}{\rho_1} - 1 \right) \Rightarrow P_2 - P_1 = \rho_2 u_2^2 \left(\frac{\rho_2 - \rho_1}{\rho_1} \right)$$

$$u_2^2 = \frac{(P_2 - P_1)}{(\rho_2 - \rho_1)} \left(\frac{\rho_1}{\rho_2} \right) \quad \text{— (5)}$$

Sub (3) in (4)

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 \frac{\rho_1^2 u_1^2}{\rho_2^2}$$

$$P_2 - P_1 = \rho_1 u_1^2 - \frac{\rho_1^2 u_1^2}{\rho_2}$$

$$= \rho_1 u_1^2 \left(1 - \frac{\rho_1}{\rho_2} \right)$$

$$P_2 - P_1 = \rho_1 u_1^2 \left(\frac{\rho_2 - \rho_1}{\rho_2} \right)$$

$$u_1^2 = \frac{(P_2 - P_1)}{(\rho_2 - \rho_1)}\left(\frac{\rho_2}{\rho_1}\right) \quad \text{— (6)}$$

By energy eqn :.

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

$$h = C_p T$$

$$h = e + \frac{P}{\rho}$$

$$e_1 + \frac{P_1}{\rho_1} + \frac{u_1^2}{2} = e_2 + \frac{P_2}{\rho_2} + \frac{u_2^2}{2} \rightarrow (7)$$

(5), (6) in (7)

$$e_1 + \frac{P_1}{\rho_1} + \frac{1}{2}\left(\frac{P_2 - P_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_2}{\rho_1}\right) = e_2 + \frac{P_2}{\rho_2} + \frac{1}{2}\left(\frac{P_2 - P_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2}\right)$$

$$e_2 - e_1 = \frac{P_1}{\rho_1} + \frac{1}{2}\left(\frac{P_2 - P_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_2}{\rho_1}\right) - \frac{P_2}{\rho_2} - \frac{1}{2}\left(\frac{P_2 - P_1}{\rho_2 - \rho_1}\right)\left(\frac{\rho_1}{\rho_2}\right)$$

$$= \frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} + \frac{1}{2}\left(\frac{P_2 - P_1}{\rho_2 - \rho_1}\right)\left[\frac{\rho_2}{\rho_1} - \frac{\rho_1}{\rho_2}\right]$$

$$e_2 - e_1 = \frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} + \frac{1}{2}\frac{(P_2 - P_1)}{(\rho_2 - \rho_1)}\left(\frac{\rho_2^2 - \rho_1^2}{\rho_1 \rho_2}\right)$$

$$= \frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} + \frac{1}{2}\frac{(P_2 - P_1)}{(\rho_2 - \rho_1)}\frac{(\rho_2 + \rho_1)(\rho_2 - \rho_1)}{\rho_1 \rho_2}$$

$$e_2 - e_1 = \frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} + \frac{1}{2}\frac{(P_2 - P_1)(\rho_2 + \rho_1)}{\rho_1 \rho_2}$$

$$e_2 - e_1 = \frac{\rho_2 P_1 - \rho_1 P_2}{\rho_1 \rho_2} + \frac{1}{2}\frac{\rho_2 P_2 + \rho_1 P_2 - \rho_2 P_1 - \rho_1 P_1}{\rho_1 \rho_2}$$

$$e_2 - e_1 = \frac{2\rho_2 P_1 - 2\rho_1 P_2 + \rho_2 P_2 + \rho_1 P_2 - \rho_2 P_1 - \rho_1 P_1}{2\rho_1 \rho_2}$$

$$e_2 - e_1 = \frac{\rho_2 P_1 - \rho_1 P_2 + \rho_2 P_2 - \rho_1 P_1}{2\rho_1 \rho_2}$$

$$e_2 - e_1 = \frac{\rho_2 P_1}{2\rho_1 \rho_2} - \frac{\rho_1 P_2}{2\rho_1 \rho_2} + \frac{\rho_2 P_2}{2\rho_1 \rho_2} - \frac{\rho_1 P_1}{2\rho_1 \rho_2}$$

$$e_2 - e_1 = \frac{P_1}{2\rho_1} - \frac{P_2}{2\rho_2} + \frac{P_2}{2\rho_1} - \frac{P_1}{2\rho_2}$$

$$e_2 - e_1 = \left(\frac{P_1 + P_2}{2\rho_1}\right) - \left(\frac{P_1 + P_2}{2\rho_2}\right)$$

$$e_2 - e_1 = \frac{P_1 + P_2}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \rightarrow (8)$$

$$e_2 - e_1 = \frac{P_1 + P_2}{2}\left(v_1 - v_2\right) \rightarrow (9)$$

Specific volume $v = \frac{1}{\rho}$

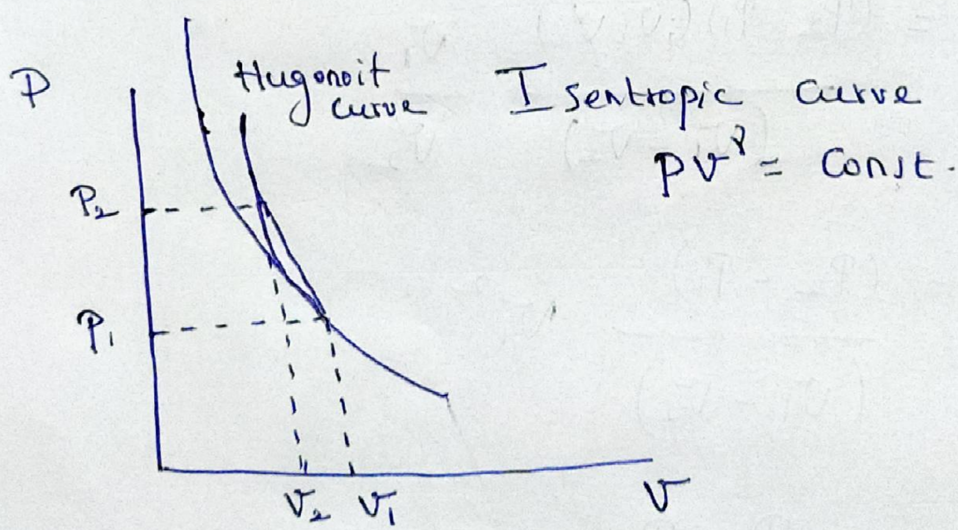
Eqn (8) & (9) $\Rightarrow$ Rankine - Hugoniot eqn.

Rankine – Hugonoit curve:

$$P_2 = f(P_1, v_1, v_2) \text{ from } (9)$$
↳ (10)

For given P_1 and v_1 the eqn (10) represents P_2 as a function of v_2.

A plot of this relation on Pv diagram is called Hugonoit – curve, it is the locus of all possible pressure – Volume conditions behind normal shocks of different strengths for a given set of upstream values of P_1 and v_1.



We know that,

$$u_1^2 = \frac{P_2 - P_1}{\rho_2 - \rho_1}\left(\frac{\rho_2}{\rho_1}\right)$$

$$= \frac{P_2 - P_1}{\frac{1}{v_2} - \frac{1}{v_1}}\left(\frac{\frac{1}{v_2}}{\frac{1}{v_1}}\right)$$

$$= \frac{P_2 - P_1}{\frac{v_1 - v_2}{v_1 v_2}}\left(\frac{v_1}{v_2}\right)$$

$$u_1^2 = \frac{(P_2 - P_1)(v_1 \cancel{v_2})}{(v_1 - v_2)} \frac{v_1}{\cancel{v_2}}$$

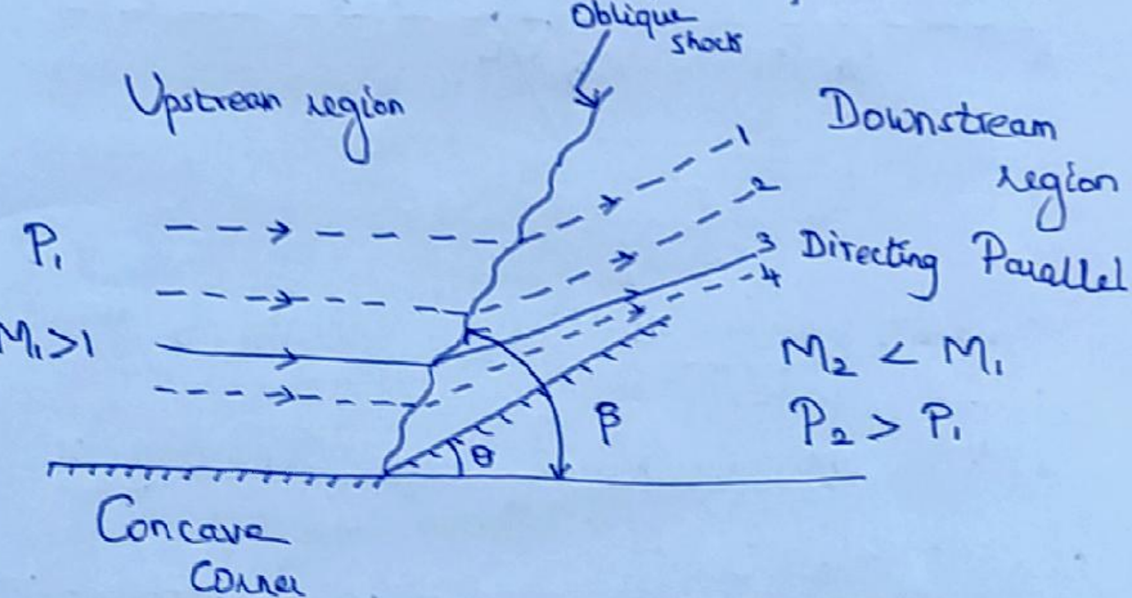
$$u_1^2 = \frac{(P_2 - P_1)}{(v_1 - v_2)} v_1^2$$

$$\frac{u_1^2}{v_1^2} = \frac{P_2 - P_2}{v_1 - v_2} \rightarrow (11)$$

Oblique Shocks :

A Compression waves occurs, inclined at an angle of flow, Such a wave is called oblique shock.

When the supersonic flow is comes across a Concave corner, an oblique shock is formed.

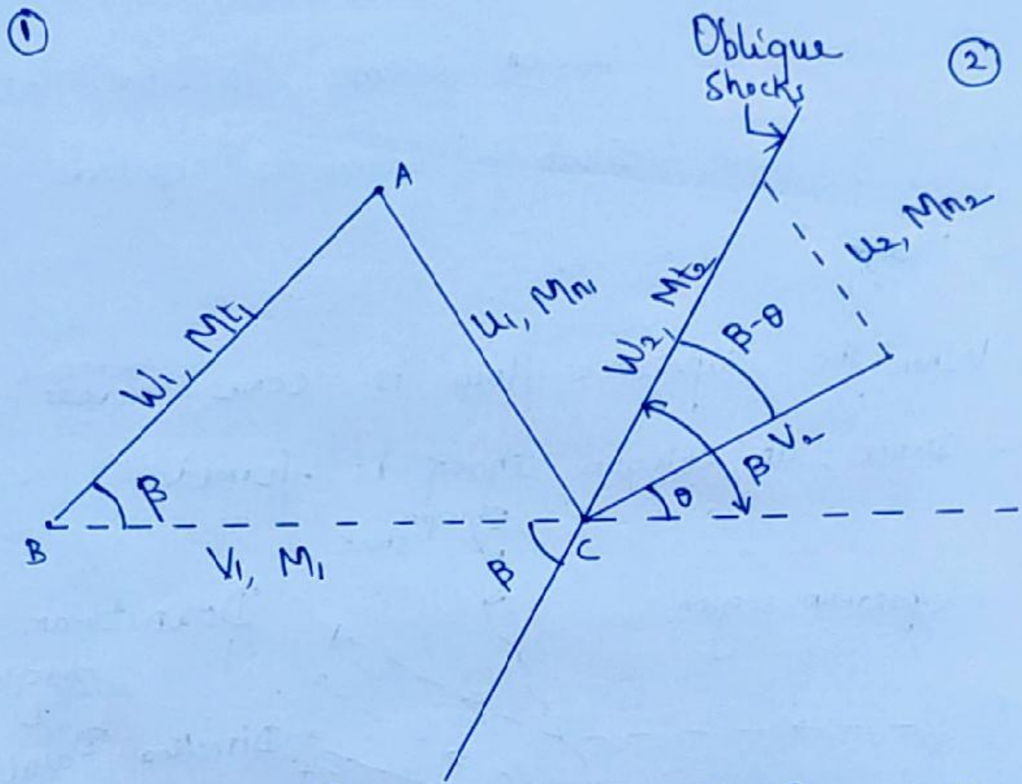


β → shock angle (or) Wave angle

θ → Wedge angle (or) flow deflection angle

* After the shock, Supersonic may decrease

* All the strean lines in upstream region of the shock wave are turned parallel to the downstream region.



Where, V_1 - is upstream velocity of the oblique shock.

V_2 - is downstream velocity of Oblique shock.

u_1 - is component of V_1 perpendicular to the shock.

u_2 - is component of V_2 perpendicular to the shock.

W_1 - is component of V_1 Parallel to the shock.

W_2 - is component of V_2 parallel to to the shock.

M_1 - is upstream mach Number oblique shock.

M_2 - is downstream mach Number oblique shock.

β - is Wave angle.

θ - is the flow deflection angle behind the shock.

M_{n1} - is normal mach number ahead of the shock

M_{n2} - is normal mach number behind the Shock.

M_{t1} - is tangential Mach number ahead of the shock.

M_{t2} - is tangential mach number behind the shock.

By Continuity equation

$$\rho_1 A_1 u_1 = \rho_2 A_2 u_2$$

$$A_1 = A_2$$

$$\rho_1 u_1 = \rho_2 u_2 \quad - ①$$

$$\rho_1 u_1 w_1 = \rho_2 u_2 w_2 \quad - ②$$

$$W_1 = W_2$$

By momentum equation:

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \quad - (3)$$

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \quad - (4)$$

From the fig,

By pythogoras theorem

$$V_1^2 = u_1^2 + w_1^2 \quad - (5)$$

$$V_2^2 = u_2^2 + w_2^2 \quad - (6)$$

sub (5) & (6) in (4)

$$h_1 + \frac{u_1^2 + w_1^2}{2} = h_2 + \frac{u_2^2 + w_2^2}{2}$$

$$w_1 = w_2$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \quad - (7)$$

From fig, ΔABC:

$$\sin\beta = \frac{M_{n1}}{M_1}$$

$$M_{n1} = M_1 \sin\beta \quad - (8)$$

WKT,

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_{n_1}^2}{2+(\gamma-1)M_{n_1}^2} \quad \text{--- (9)}$$

(8) in (9)

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2 \sin^2\beta}{2+(\gamma-1) M_1^2 \sin^2\beta} \quad \text{--- (10)}$$

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{(\gamma+1)} (M_{n_1}^2 - 1) \quad \text{--- (11)}$$

(8) in (11)

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{(\gamma+1)} (M_1^2 \sin^2\beta - 1) \quad \text{--- (12)}$$

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)\left(\frac{\rho_1}{\rho_2}\right) \rightarrow (13)$$

(10), (12) in (13)

$$\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{(\gamma+1)} (M_1^2 \sin^2\beta - 1)\right]\left[\frac{2+(\gamma-1) M_1^2 \sin^2\beta}{(\gamma+1) M_1^2 \sin^2\beta}\right]$$

$$\frac{T_2}{T_1} = \left[\frac{(\gamma+1) + 2\gamma (M_1^2 \sin^2\beta - 1)}{(\gamma+1)}\right]\left[\frac{2+(\gamma-1) M_1^2 \sin^2\beta}{(\gamma+1) M_1^2 \sin^2\beta}\right]$$

$\hookrightarrow$ (14)

From fig,

$$\tan\beta = \frac{u_1}{w_1}$$

$$\tan(\beta-\theta) = \frac{u_2}{w_2}$$

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{\frac{u_2}{w_2}}{\frac{u_1}{w_1}} \quad (\because w_1 = w_2)$$

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} \quad \text{— (15)}$$

(10) in (15)

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

$$\frac{\tan\beta-\tan\theta}{(1+\tan\beta\tan\theta)\tan\beta} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}$$

$$\frac{\tan\beta-\tan\theta}{\tan\beta+\tan^2\beta\tan\theta} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}$$

$$\tan\beta - \tan\theta = \tan\beta + \tan^2\beta\tan\theta\left[\frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}\right]$$

$$\tan\beta - \tan\theta = \tan\beta\left[\frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}\right] + \tan^2\beta\tan\theta\left[\frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}\right]$$

$$\tan\beta - \tan\beta\left[\frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}\right] = \tan^2\beta\tan\theta\left[\frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}\right] + \tan\theta$$

$$\tan\beta\left[1 - \left[\frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}\right]\right] = \tan\theta\left\{\tan^2\beta\frac{\left[2+(\gamma-1)M_1^2\sin^2\beta\right]}{(\gamma+1)M_1^2\sin^2\beta} + 1\right\}$$

$$\tan\beta\left[\frac{(\gamma+1)M_1^2\sin^2\beta - 2 - (\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)\cancel{M_1^2}\sin^2\beta}\right]$$

$$= \tan\theta\left\{\frac{\tan^2\beta\left[2+(\gamma-1)M_1^2\sin^2\beta\right] + (\gamma+1)M_1^2\sin^2\beta}{(\gamma+1)\cancel{M_1^2\sin^2\beta})}\right\}$$

$$\tan\beta\left[\gamma M_1^2\cancel{\sin^2\beta} + M_1^2\sin^2\beta - 2 - \gamma\cancel{M_1^2}\sin^2\beta + M_1^2\sin^2\beta\right]$$

$$= \tan\theta\left[\frac{\sin^2\beta}{\cos^2\beta}\left[2+(\gamma-1)M_1^2\sin^2\beta\right] + (\gamma+1)M_1^2\sin^2\beta\right]$$

$$\tan\beta\left[2M_1^2\sin^2\beta-2\right]$$

$$=\tan\theta\left\{\frac{\sin^2\beta\left[2+(\gamma-1)M_1^2\sin^2\beta\right]+(\gamma+1)M_1^2\sin^2\beta\cos^2\beta}{\cos^2\beta}\right\}$$

$$2\tan\beta\left[M_1^2\sin^2\beta-1\right]$$

$$=\tan\theta\left[\frac{2\sin^2\beta+(\gamma-1)M_1^2\sin^4\beta+(\gamma+1)M_1^2\sin^2\beta\cos^2\beta}{\cos^2\beta}\right]$$

$$=\tan\theta\left[\frac{2\sin^2\beta+\sin^2\beta\left[(\gamma-1)M_1^2\sin^2\beta+(\gamma+1)M_1^2\cos^2\beta\right]}{\cos^2\beta}\right]$$

$$2\tan\beta\cos^2\beta\left[M_1^2\sin^2\beta-1\right]$$

$$=\tan\theta\left[2\sin^2\beta+\sin^2\beta\left[\gamma M_1^2\sin^2\beta-M_1^2\sin^2\beta+\gamma M_1^2\cos^2\beta+M_1^2\cos^2\beta\right]\right]$$

$$2\times\frac{\sin\beta}{\cos\beta}\cos^2\beta\left[M_1^2\sin^2\beta-1\right]$$

$$=\tan\theta\left\{2\sin^2\beta+\sin^2\beta\left[\gamma M_1^2(\sin^2\beta+\cos^2\beta)+M_1^2(\cos^2\beta-\sin^2\beta)\right]\right\}$$

$$2\sin\beta\cos\beta\left[M_1^2\sin^2\beta-1\right] = \tan\theta\left\{2\sin^2\beta+\sin^2\beta\left[\gamma M_1^2+M_1^2\cos2\beta\right]\right\}$$

$$2\sin\beta\cos\beta\left[M_1^2\sin^2\beta-1\right] = \tan\theta\left\{2\sin^2\beta+M_1^2\sin^2\beta\,(\gamma+\cos2\beta)\right\}$$

$$\tan\theta = \frac{2\sin\beta\cos\beta\,(M_1^2\sin^2\beta-1)}{2\sin^2\beta+M_1^2\sin^2\beta\,(\gamma+\cos2\beta)}$$

$$\tan\theta = \frac{2\sin\beta\cos\beta\,(M_1^2\sin^2\beta-1)}{\sin^2\beta\left[2+M_1^2(\gamma+\cos2\beta)\right]}$$

$$\tan\theta = \frac{2\cot\beta\,(M_1^2\sin^2\beta-1)}{M_1^2(\gamma+\cos2\beta)+2} \quad \text{— (16)}$$

Equation (16) is known as θ-β-M relation.

Case (i) : $\theta > \theta_1$

For any upstream machnumber, of the shock, there is a maximum value of θ at given mach no. M_1, if $\theta > \theta_1$, there is no solution for straight oblique shock.

Case (ii) : $\theta < \theta_1$

- There are two possible solutions, for each value of θ and M_1 having two different wave angles.

- The large value of β is called strong shock solution and small value is weak solution.

- For strong shock solution : flow behind the shock becomes subsonic

- For weak shock solution :. Flow remains supersonic, except for a small range of θ values slightly smaller than θ_{max}

Case (iii) :·· if $\theta = 0$, $\beta = 90°$, normal shock appears. if β decreases to the limiting value, (i.e) shock disappears and only Mach waves prevail in the flow field.